Ideals: ~~definitions~~

Definition:- Let $(R,+,\cdot)$ be a ring. Then $(S,+,\cdot)$ is an ideal of $(R,+,\cdot)$ if and only if

(i) $a, b \in S \Rightarrow a+(-b) \in S$

(ii) $s \in S$ and $r \in R \Rightarrow rs \in S$ and $sr \in S$ $\forall\, r \in R$ and $\forall\, s \in S$

- Left ideal:- A non-empty subset S of a ring R is said to be a left ideal of R if (i) S is a subgroup of R w.r.to addition
  (ii) $rs \in S$ $\forall\, r \in R$ and $\forall\, s \in S$

- Right ideal:- A non-empty subset S of a ring R is said to be a right ideal of R if
  (i) S is a subgroup of R under addition
  (ii) $sr \in S$ $\forall\, r \in R$ and $\forall\, s \in S$

Theorem:- Let $(R,+,\cdot)$ be a ring and $(S,+,\cdot)$ be an ideal in $(R,+,\cdot)$. Then $(S,+,\cdot)$ is a subring of R i.e. every ideal is a subring.

Proof:-

Given $(S,+,\cdot)$ be an ideal in $(R,+,\cdot)$ i.e. it is given that

(i) $a, b \in S \Rightarrow a+(-b) \in S$

(ii) $a \in S$, $r \in R \Rightarrow ar \in S$ and $ra \in S$

It is ~~order~~ to prove that $(S,+,\cdot)$ is a subring of R i.e. it is to prove that

ⓐ $a, b \in S \Rightarrow a+(-b) \in S$

ⓑ $a, b \in S \Rightarrow ab \in S$

It is obvious that (a) is the same as form (i).
What we have to prove in fact is (b).
We do this as follows:
Let $a \in S$, $b \in S$ then $b \in R$ since S is a subset of R.
Hence by (ii) $ab \in S$
Thus (b) is proved and hence the theorem.

But the converse is not true
i.e. there are subrings which are not ideals.
Some examples are given below:
① The set Q of rational numbers with the usual operations of addition and multiplication is a commutative ring.
Thus,
the set I of integers is a subset of Q and is a commutative ring.

Hence, I is a subring of Q. But I is not an ideal of Q.
Because we can find an integer $a \in I$ and one rational number $r \in Q$ such that $ar \notin I$.
For example,
take $a = 1$ and $r = \frac{1}{2}$
so, that $ar = 1 \cdot \frac{1}{2} = \frac{1}{2} \notin I$.

Theorem :- Prove that intersection of two ideals in a ring is also an ideal in the ring.

Proof:-

Let S_1 and S_2 be two ideals in a ring R.

Let $S = S_1 \cap S_2$

Since every ideal is a Subring

Therefore, each of S_1 and S_2 is a Subring.

We know that the intersection of two Subrings is a Subring.

Therefore, $S_1 \cap S_2 = S$ is a Subring.

Now, in order to prove that S is an ideal.

We need to prove that $a \in S, r \in R \Rightarrow ar \in S$.

Since; $a \in S$ $\therefore$ $a \in S_1$ and $a \in S_2$

Also, $a \in S_1$ and $r \in R \Rightarrow ar \in S_1$ $\{\because S_1$ is an ideal$\}$

and similarly, $a \in S_2$ and $r \in R \Rightarrow ar \in S_2$

Hence, $ar \in S_1 \cap S_2 \Rightarrow ar \in S$

Hence, S is an ideal in R.